The Poincaré Models of the Hyperbolic Plane Plane*

George Francis†

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1 Conformal Models

The Beltrami-Klein model of the hyperbolic plane is not as useful for measuring angles as it is for measuring distances. We therefore consider the *conformal models* of Henri Poincaré. These are called conformal because their protractor coincides with the Euclidean protractor. Thus angles are 'true'. For *projective models*, like the Beltrami-Klein model, straight lines are straight also in the Euclidean sense. This is not the case in the conformal models.

There is a deeper geometrical difference between the two classes of models, the conformal and the projective ones. This will be explained in detail later. For now, suffice it to say that there are many more models of the hyperbolic plane and there is a way of relating them all to each other by various projections in 3-space. The two conformal models we shall study are the *Poincaré disk model* and the *Poincaré upper half plane model.* Although the second is in many ways easier to work with the former is more easily related to what we've have been doing in the Beltarami-Klein disk model. So we begin with that.

As before, we shall introduce some geometrical machinery which by itself has nothing to do with hyperbolic geometry. It is useful and interesting for its own sake. It happens that geometers found it useful for building models of the hyperbolic plane.

Interpretation of Primitives.

In the *Poincaré disk model* the P-points are again the points inside a circle. The

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†Prof. George K. Francis, Mathematics Department, University of Illinois, 1409 W. Green St., Urbana, IL, 61801.

points on the Poincaré circle are not P-points themselves. But we think of them as being "at infinity"; of being at either "end" of the infinite P-lines.

For P-lines we shall use circular arcs which are perpendicular to the Poincaré-circle. There are no such arcs through the center (why?). So we shall adopt the diameters as P-lines too.

2 Inversive Geometry of the Plane

In order to study the conformal models further it is useful to know a few things about the geometry of circles and their reflections in lines and other circles. This too is non-Euclidean geometry, and has its independent practicality. It plays the same role for the conformal models of the hyperbolic plane, as the geometry of perspectivities did in our cursory study of the projective model.

In the *inversive plane* we treat straight lines and circles as members of the same species, which we shall call *generalized circles* when it is necessary to remind you of this convention. We shall also treat *infinity* as a single, ideal point[1]

Given a circle C with center at O and radius r in the plane, the *inverse* of a point P in C is defined to be that point Q on the ray OP so that the product of the distances

$$OP \cdot OQ = r^2$$

.

If you extend a radius OR to P then the chord from the two tangents from P to the circle cuts the radius at the inverse of P. If P lies inside the circle, its inverse lies outside, and the roles have been reversed. Recall the construction of pole and polar in the discussion of the Beltrami-Klein model of the hyperbolic plane.

There are a few special cases to consider. When P lies on C, then it is its own inverse. The inverse of the center of C is the point at infinity ∞. We must also say what it means to invert a point in the honorary circles, the straight lines.

If C is a straight line, its center is said to be at infinity, but that doesn't help much, because the radius is infinite too. So here we define the inverse of P to be the *reflection* in the line, i.e. if PQ crosses the line at R then we want

[1] The various ways geometers treat 'infinity' has no philosophical implications. It is a formal expression of what it means, in a particular context, when things happen 'far, far away". In the context of perspective painting of the Renaissance, lines which are in reality parallel, focus at one point on the horizon in a perspective drawing. Therefore, in the projective plane there is one ideal line at infinity, and an ideal point on the ideal line is represented by any one of a class of mutually parellel lines. In the hyperbolic plane, there is also an ideal line, but it is not part of the hyperbolic plane. In the Klein model, the ideal points are the points on the boundary circle. Inversive geometry establishes a single ideal point at infinity.

$PR = RQ$. It is also appropriate to call the inversion of points in a round circle a 'reflection' in the the circle, even though calling it an 'inversion' is historically more consistent.

Once again, to pursue circle inversion synthetically any further is difficult and we have recourse to analytic geometry to simplify our study of it.

2.1 Equation of a Circle.

It is interesting that we can write down one equation which includes both circle and straight lines.

$$k(x^2 + y^2) + ax + by + c = 0 \tag{1}$$

Note that when $k = 0$, we have the most general equation of a straight line. Now assume $k > 0$, since we can always arrange the equation to have this form (why?). Now complete the square as you were taught to do in high-school.

$$x^2 + 2\frac{a}{2k}x + (\frac{a}{2k})^2 + y^2 + 2\frac{b}{2k}y + (\frac{b}{2k})^2 = \frac{a^2 + b^2 - 4ck}{4k^2} \tag{2}$$

From which is follows that we have an equation of a circle centered at $(-\frac{a}{2k}, -\frac{b}{2k})$ and with radius equal to $\frac{\sqrt{a^2+b^2-4ck}}{2k}$, provided that the expression inside the sqare-root is positive, of course.

Question 1 Work your way through these algebraic computations until you have memorized them. The need for similar computations arises often, especially on exams.

Next, compute the inverse of a point (x, y) in the unit circle to be $(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. The power of algebra is beautifully illustrated by our proof of the following.

Theorem 1

Inversion in a circle C takes a generalized-circle to a generalized-circle.

Proof. The case that C is actually a straight line is uninteresting, for then this theorem just says that reflections in lines takes lines to line and circles to circles. This might be difficult for you to prove, but it it something that could have been assigned for homework in high-school.

Here is our strategy for the interesting and surprising case. First, use a coordinate system that makes C into the unit circle. What we plan to show is that the inverses of the points (x, y) that satisfy the equation of a circle, also satisfy an equation of a circle, which turns out to be closely related to the given one.

Lemma 1. If you divide through the equation (1) of a circle by $x^2 + y^2$ you obtain this equation

$$k + a\frac{x}{x^2+y^2} + b\frac{y}{x^2+y^2} + c((\frac{x}{x^2+y^2})^2 + (\frac{y}{x^2+y^2})^2) = 0 \tag{3}$$

Question 2 Simplify the expression multiplied by c and, thus, prove the lemma.

This says that the inverse of (x, y) satisfies the equation of a circle which is identical with the one satisfied by (x, y), except that the role of k and c have been swapped.

Question 3 Given a circle of radius r centered at (p, q). Compute the radius and center of its inversion in the unit circle. Hint: Lemma 1 tells you its equation. Now have another look at Equation 2.

Question 4 Show that the inversion of a generalized circle which happens to be a straight line, is a circle which passes through the origin of the mirroring circle.

We are now ready to tackle the really hard theorem that says that circle inversion is a conformal (angle-preserving) mapping of the plane (extended by one point at infinity.)

First we need the expression of the angle between two circles

$$k_i(x^2 + y^2) + a_i x + b_i y + c_i = 0,\ i = 1, 2$$

at a common point, (x_0, y_0). Now is a good time to remember what you learned in the calculus.

Lemma 2. To compute the tangent line to the curve $f(x, y) = 0$ at a point (x_0, y_0) on it, take the differential of this equation

$$0 = d0 = df(x, y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

substitute (x_0, y_0) into the partials and replace dx by $(x-x_0)$ and dy by $(y-y_0)$. The coefficients are the components of the normal vector to the curve at the point.

Now the angle between two crossing curves is just the angle between their (not necessarily unit) normals at that point, which are:

$$(2k_1x_0 + a_1, 2k_1y_0 + b_1) \quad , \quad (2k_2x_0 + a_2, 2k_2y_0 + b_2) \tag{4}$$

Hence the cosine of the angle between them is

$$\text{cosine} = \frac{a_1a_2 + b_1b_2 - 2k_1c_2 - 2k_2c_1}{\sqrt{a_1^2 + b_1^2 - 4k_1c_1}\sqrt{a_1^2 + b_1^2 - 4k_1c_1}}. \tag{5}$$

Question 5 Verify this formula as elegantly as you can. Recall the definition of the dot product of two vectors, and solve it for the cosine of the angle between them.

Recall from (3) that the only difference between the equations of a circle and its inverse is that the roles of k and c are exchanged. If $k(x^2 + y^2) + ax + by + c = 0$ is the equation of the circle, then the equation of its inverse is $c(x^2 + y^2) + ax + by + k = 0$. But, look at the expression (5) for the cosine of the angle between two circles. It does not change if you swap k and c. This proves

Theorem 2. Circle inversion is conformal.

Question 6 Actually, we haven't quite proved this theorem. There remains the case that the mirror is not the unit circle. If the mirror is another circle, we can always arrange our Cartesian coordinate system so that it's center is the origin, and our scale makes it a unit circle. But then when the mirror is a straight line, the best we can do is choose a coordinate system for which the mirror becomes the y-axis. Now mimic the above proof to show that reflection in lines is conformal.

3 The Poincare Disk Model

As in the the Klein-Beltrami model, we interpret a "point" as a point inside the unit disk in the plane. Unlike the K-model, "straight lines" will now be interpreted as circular arcs inside the Poincaré-disk which are perpendicular to the unit circle.

Lemma 3. The circles perpendicular to the unit circle are those whose equation has this form

$$k(x^2 + y^2) + ax + by + k = 0. \tag{6}$$

Proof. Apply the formula for the cosine of the angle of intersection of two circles. (Yes, really do it!)

Note that for $k = 0$ we are talking about a diameter of the Poincaré-disk, and the center of the Poincaré-disk becomes a distinguished point. This is, however, a feature of the model, not of the hyperbolic plane. To an insider, everywhere in the hyperbolic plane looks the same, just as is the case for us in the Euclidean plane.

We can underscore this situation by taking the *transformational* approach. We continue interpretating geometrical primitives, such as congruence. Define an elementary *P-isometry*, a.k.a. *congruence* to a *reflection* in a P-line. And such a P-reflection shall be an inversion in the circle that the P-line lies on. Remember, the P-line is only that portion of a generalized circle perpendicular to the Poincaré-cirlce which is *inside* the disk.

A general *P-isometry* is defined to be a succession of P-reflections. Note the similarity of this definition with definition of a K-congruence to be a succession of perspectivities. But instead of comparing two line seqments, or two figures made up of line segments by their congruence, we can construct the image of every point simultaneously. We don't have to define P-congruence separately, but simply say that two figures are P-congruent if there is a P-isometery taking one to the other. This way, we have (finally) made precise what Euclid had in mind all along.

Theorem 3. This is an appropriate interpretation of congruence.

Proof. Strictly speaking, to check that an interpretation of "congruence" is OK you need to check that the axioms in the formal system associated with this notion are true under this interpretation. Thus, if we took the formal approach, using Hilbert's axioms, or the SMSG axioms, we would have to check that SAS is true, for example. In our informal approach, we content ourselves in checking just a few "obvious" properties an isometry should have.

Lemma 4. P-reflection in a P-line, is a one-to-one transformation of the P-plane onto itself.

Proof. You need to check that an inversion in a generalized circle (the *mirror*) which is perpendicular to the unit circle, takes the P-points to P-points, and the P-lines to P-lines. Only the latter needs a second thought.

Since inversions are conformal, in that they maintain the Euclidean angle measure, it is reasonable to interpret P-angle to mean the E-angle between the two P-rays. In particular, Euclid's fourth postulate "all right angles are equal" makes a little more sense in this context.

So, while angle measurement is easier in this model, some simple things, like Euclid's 1st Postulate become much harder. Here, however, is a very useful theorem. It says, in effect, that we may always move our "work" to the center of the Poincaré-disk.

Theorem 4. For every P-ray there is an isometry that moves the tail of the ray to the center of the Poincaré-disk, and the ray itself to a radius of the Poincaré-disk.

Proof. This is, of course, an exercise in inversive geometry. We give it here to impress you with the power of inversive geometry.

Step A Consider the case that the given P-ray is not already on a diameter[2] and so lies on a "round" circle L.

1. Locate the diameter D parallel to the chord determined by L.
2. Extend the radius perpendicular to D through the center of L to its intersection Q with L on the outside of the Poincaré-disk.
3. Draw the tangents from Q to the Poincaré-disk so you can draw a circle M centered on Q and perpendicular to the unit circle. This is the *mirror* of a P-reflection.
4. Note that, if you were to invert L in M you would get D.(Why?).

Step B Thus there really is a P-isometry that takes any old P-line to a diameter. But we can do more. We can move an arbitrary P-point on a P-line to any other P-point on it by a P-reflection. We do this for the special case that the baseline is a diameter.

1. Given a point N on a diameter D of the unit disk, find its inverse P.
2. From P draw the circle perpendicular to the unit circle. This is your mirror.
3. Now, if you were to reflect Q in this mirror it would end up at the center of the unit circle. (Why?). Of course, it takes the diameter $D4$ to itself.(Why?).

Question 7 There are a lot of 'why's' to be collected. Do that, but don't forget to use "plain-speak", i.e. use English sentences that tell the reader what you are doing. This is major assignment. When there is a complicated argument to be learned, you must write your way through it "in your own words," as your English teacher used to say in high school.

4 Constructions in the P-Model.

To illustrate Theorem 4, we shall use it to show that P-circles are also E-circles.

[2] That case is part of Step B.

Corollary 1. A P-circle is also an E-circle, though the E-center of the circle is closer to the center of the Poincaré disk than its P-center.

Proof. Note that this also says that if the P-center of the given circle is the center C of the Poincaré-disk, its two centers coincide. Otherwise, they at least lie on a common radius of the Poincaré-disk.

Given a P-radius, QS, we know from Theorem 4 how to construct an isometry which takes it to CR, where C is the center of the Poincaré-disk, and R is a point on a radius. If a P-circle with center at C is an E-circle, then the same isometry reversed takes it to an E-circle. So we still need:

Lemma 5. A P-rotation about the center of the Poincaré-disk is an E-rotation.

Proof of the lemma. We can think of a rotation of the radius CR as a series of P-reflections in other P-lines through the same point, C. But at the center, these P-lines are diameters, and so P-reflections in them are Euclidean. So, the object must so obtained must be a Euclidean circle. □

While inversions takes circles to circles, they don't take their centers to each other. Your intuition suffices to guess the location of the P-center. By symmetry it should be on the same diameter of the Poincaré-disk as the E-center. Equal P-lengths must get E-shorter as you approach "infinity". The

Now let us identify the straightedge and the compass in the P-model. By the *P-straightedge* we mean a construction that draws the P-line through two given P-points, A, B. If these two points are on a diameter of the Poincaré-disk, the construction is that diameter. Otherwise, we do know an entire line on which the center of the circle through A, B lies. (How?) We are seeking that circle throug A, B which is also perpendicular to the unit circle. If we can find a third point that must lie on this circle we can locate its center, and we're done. But the inverse A' of A in the unit circle is such a point, because circles perpendicular to the unit circle are invariant under inversion.

Question 8 Identify the construction steps in the diagram, and perform a few more straightedge constructions.

In what follows we shall frequently need the following construction:

Perpendicular Construction. Given a P-point Q, find the P-line perpendicular to the radius CP, where C is the center of the Poincaré-disk.

Solution. The required P-line is a circle through Q which is perpendicular to the unit circle. Since it is invariant under inversion in the unit circle, the inverse Q' is on it. But then QQ' is a diameter of the required circle, and we know how to complete the construction. (Don't we?)

Circle Construction Given a P-radius QR, to find the P-circle.

Solution. We know one point on the required circle. If we could locate a diametrically opposite point we'd be done. Suppose first (the easy case) that QR is on a diameter of the Poincaré-disk. Then construct the P-perpendicular this diameter at Q as a P-mirror and invert R in it to R'. Since this is a P-isometry (why?) RR' must be a P-diameter. It must also be an E-diameter (why?). So we can bisect it to find the E-center and we're done.

Finally, suppose QR are not on a diameter. Then they're on a P-line which is a circular arc perpendicular to the unit circle. Let's try to do it the "same way" and patch the argument when needed. Extend CQ to a radius. We know that the center K of the required circle lies on that line. As before, draw the mirror through Q perpendicular to CQ and P-reflect (i.e. invert) R in it to R'. All we know about RR' is that it is a chord of the required circle. That gives us a second line on which the center must lie and we're done.(Why?).

Conclusion, for now. So, having a compass and straightedge, and a notion of congruence, we can, in principle, do the constructions in Euclid and elsewhere that belong to neutral geometry. We can copy a line segment to a ray (use Theorem 4 twice). The perpendicular bisector of a line segment can be constructed with two equilateral triangles. You could, with considerable effort, demonstrate the gap left in the Exterior Angle Theorem construction. That would demonstrate that the Poincare geometry is hyperbolic, though simpler violations of Euclid's Fifth Postulate are constructible. Since it is a conformal model, constructing right angles are no problem and the Saccheri Railroad is in sight. Particularly instructive would be your discovery that the rails are now circular arcs which are *not* perpendicular to the unit circle. What happens to the Saccheri RR at infinity is not ambiguous, there is a definite angle at which the two rails meet the centerline.

However, the constructions become very difficult here, mainly because it is always difficult to construct the Euclidean centers and radii of the P-lines. There is another model, also due to Poincaré, which is much easier to work with. We study this next.

5 The Upper Half Plane Model

Here we take the points above the x-axis as our H-points[3] and the H-lines shall be on generalized circles which are perpendicular to the horizontal axis, the *baseline* of the upper half plane, for short.

Question 9 Show analytically that the H-lines are either straight verticals or semicircles with centers on the baseline.

[3] We have run out of prefix letters if P- stands for Poincaré disk, K- for the Klein disk let's use H- for the upper **H**alf plane.

Question 10 Find the mirror for which the unit-disk inverts to the upper half plane. Hint, the center of the mirror must lie on the unit circle and pass through the points that the x-axis and unit-circle have in common. See the picture for a hint.

There are significaant consequences of the theorem you proved in the last exercise. Since the Poincaré disk model and the UHP model differ only by an inversion, the constructions and relations which are "true" in one model, and which remain true under inversion, are true in the other model. Thus H-circles must also be Euclidean, because we have seen that P-circles are so, and circles go to circles under inversions. Such Euclidean attributes like being a diameter of the P-disk, or a vertical of the H-plane are not preserved under inversions.

Question 11 Under the inversion of Problem 9. find the H-point corresponding to the center of the P-circle. Draw a number of P-lines corresponding to the verticals of the H-plane. In the H-plane, what is "at infinity" ?

The fundamental constructions are easier to perform in the H-plane because the centers of the H-lines are all on the baseline (except for the verticals, of course.) To illustrate this, we construct the *H-inch-worm.*

Question 12 Do the following constructions for the special H-lines, the verticals.

Step A Given non-vertical points A, B, the perpendicular bisector of AB must pass through the center of every circle through A, B. The perpendicular bisector crosses the baseline at the center of that circle which is also perpendicular to the baseline. This is the H-line through A, B.

Step B To double the segment PQ, we need a mirror m at one end and perpendicular to it. To draw the mirror find the center on the baseline of the semicircle through P, Q. Use it to draw its tangent from Q. Follow this tangent to the baseline to find the center M of the mirror. Draw the mirror m with your compass. Now, the inverse R of P in m must simultaneously line on the radius of m and of the H-line PQ extended. Your ruler will find this point.

Question 13 Repeat this construction in both directions to construct an "inchworm" that marks the integers on the line with unit segment PQ.

Step C We must also be able to *bisect* a given segment in the H-model. Note that the construction in Step B can be done in reverse. Start with segment AB on an H-line ℓ. Extend the chord of the semicircle ℓ to the baseline. That must be the center of the mirror m, which is the perpendicular bisector of AB. A radius must be tangent to ℓ. The point of tangency is therefore, the H-midpoint of the H-segment.

6 Rulers in the H-plane

Note that "doubling" and "halving" is all the construction you need to approximate every real number on an H-line with given unit. The usual argument goes something like this. Just as every real number has a decimal expansion (finit, repeating or neither), and the decimal expansion is in fact a power series (in powers of ten) that converges to the number, so too you could use a binary expansion. Such "binimals" consist of a sequence of 0s and 1s and a binimal point.

7 Constructing H-circles

To construct an H-circle on a given radius QR is easier too. We already know that it is a Euclidean circle, and by symmetry, its E-center must lie on the vertical through its H-center Q. Now double the segment RQ to extend this H-radius to a H-diameter ending at R'. This gives us a pair of points on the same circle. The Euclidean perpendicular bisector of this chord also goes through the E-center. So find it and draw the H-circle centered at Q and H-radius QR.

Question 14 The above constructions is incomplete. There are cases for QR for which it does not work as described, and it's not just when QR is vertical. Can you fix these gaps? Do it!

Question 15 You have all the tools for repeating the Exterior Angle Theorem construction in the H-model. Do it!

8 The Saccheri Railroad in the H-plane

The final construction in the H-plane should be the Saccheri "railroad". Suppose we start with an H-line ℓ and mark its endpoints L, R on the baseline. Let r be any other circular arc through L and R. Of course, r will *not* be an H-line, because it won't be perpendicular to the baseline. Now let m be any perpendicular to ℓ and consider it a mirror. The two points L and R will be inverses of each other relative to m (why?). Therefore (think!) every other circle through L and R will also be perpendicular to the mirrror. In particular, H-reflection in the mirror will map the rail r into itself, and every "tie" into another tie. Recall that a tie is just an H-segment perpendicular to ℓ. And inversion is conformal.

Question 16 Figure out what the Saccheri railroad for a vertical H-line looks like.

Now, as a final application Problem 9 we conclude that the equidistants in the P-plane are also circles that pass through the endpoints of a P-line.