The Beltrami-Klein Model of the Hyperbolic Plane*

George Francis†

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1 Introduction

In this chapter we introduce the Beltrami-Klein model of the hyperbolic plane. This consists of a re-interpretation of the geometrical primitives, points, lines, angles etc., which differs from the intuitive, Euclidean notions. In this interpretation, Euclid's fifth postulate does not hold, and thus it becomes a model of a non-Euclidean geometrical axiom system. The interpretation occurs inside Euclidean geometry, so we can use our customary geometrical skills in drawing accurate pictures with standard tools. This pedagogically reassuring feature was promoted by Felix Klein. It's central role in the logical foundation of geometry will be discussed later.

Here we shall concentrate on discovering the features of hyperbolic geometry by working with one of its models. We defer the analytic description of this model until later. Here we first use the familiar tools of ruler, compass, 30-60-90 transparent triangle, and inkpen. Later we'll also use the computer based geometry construction set called "The Geometer's Sketchpad".

The definitions will all be in terms of Euclidean geometry, and by drawing your pictures very carefully, you can discover theorems and give valid, if heuristic, proofs by construction. We offer some exercises of varying difficulty, which you should work on until you have solved them.

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†Prof. George K. Francis, Mathematics Department, University of Illinois, 1409 W. Green St., Urbana, IL, 61801. (C)2202 Board of Trustees.

2 Interpreting the Primitives

For starters, let us interpret the primitives, *point, line, incidence*, for the Klein model as follows. The K-points shall be the points on the interior of the unit disc. The K-lines shall be the chords of this disc, excluding their end points. A K-point is K-incident to a K-line if and only if they are incident in the Euclidean sense. We shall refer to this structure as the *Kleindisk* and use the modifiers *K-points* and *K-lines*, and more rarely also *E-points* and *E-lines*, only when necessary to avoid confusion.

Note that the points on the circular rim of the Klein-disc are not among the K-points, thus K-lines have no K-endpoints. Like E-lines, K-lines are infinite. Let us now adopt Euclid's definition of *parallel* to mean *not-crossing*. Then it is immediatley clear that through a K-point not on a given K-line there are infinitely many K-lines K-parallel to the given K-line. Expressing the Euclidean parallel postulate as there being exactly one parallel to a given line through a point not on the line, is called *Playfair's postulate*.

You have already done most of this exercise in previous homework and review problems. You should put a clean solution to this question into your journal.

Question 1 That Euclid's first postulate holds in the K-geometry is pretty obvious. Why? But if we (naively) were to adopt E-congruence for K-congruence then Euclid's second postulate would be false. Show this. Again, if we tried to define K-circles as E-circles which lie entirely in the K-disc we would soon get into trouble. But this is more difficult to establish. Instead, find some theorem in Euclidean geometry which would become false in K-geometry if we were to interpret K-perpendicular as the same as E-perpendicular. So we shall interpret these geometric notions differently than you are accustomed to, so that all but the parallel postulate are true.

3 Definition of K-congruence for Segments

A fundamental construction in geometry is the transfer of line segments from here to there. More precisely, given a line segment AB and a ray ρ_C. Recall that a ray is a subset of a line ℓ through C. When you mark off a point D on ρ_C so that AB and CD are have the same length then you have *transferred* the segment to the ray.

Suppose you look at the Kleindisk K from a *viewpoint* V outside of K. Imagine putting your eye near V and very close to the plane of K. Some chords of K would appear to be the same because their endpoints would line up with the viewing rays from V. We shall say that two chords of K are *in perspective* as seen from V when they line up visually in this way.

On the other hand, it is obvious that any two chords are in perspective from some viewpoint. If they happen to be E-parallel, then this viewpoint is said to be *at infinity*, and all other sightlines from this viewpoint are parallel.

Question 2 Draw examples of the various typical ways two chords can be in perspective. Why do viewpoints always lie outside the Kleindisk? Show how two chords having one endpoint in common are in perspective for all viewpoints on the E-line extending the chord joining the other endpoints. Note that in all other cases, there only one viewpoint for non-crossing chords, and two if they cross.

Now, two K-segments shall be declared to be *K-congruent*, $AB \cong CD$, provided they are *in perspective*. In other words, extend the segments to their K-lines (Euclid's Postulate 2 holds), which are E-chords in K. Then check that everything lines up from the/a viewpoint.

Question 3 Label a picture in which two segments, AB and CD, extend to chords $A'B'$ and $C'D'$. The four points A', A, B, B' match the points C', C, D, D' as seen from a viewpoint V. If $A'D' \parallel C'D'$ the V is at infinity, and all four lines $B'D', BD, AC, A'C'$ are parallel.

This relationship between two sets of four collinear points in the Euclidean plane is called a *perspectivity*. It is related in a technical and historical way to drawing in perspective, which was an important industry in the late Renaissance. It promoted the evolution of geometry.

Question 4 Here is another way of asking the same question. Draw at least two pictures illustrating what is written here. Given segment AB on chord $A'B'$, find where, on a second chord $C'D'$, there is a segment CD K-congruent to AB. To solve this question, join A' to C' and extend this line until it meets the extension of $B'D'$ at V. From V, draw lines through A and B and extend them (if necessary) until they intersect $C'D'$ at C and D respectively. The collinear foursome $A'ABC'$ looks the same from V as does $C'CDD'$, so they are K-congruent by definition.

Note that, in Euclidean terms, the lengths of AB and CD need *not* appear to be the same even if they are K-congruent. This is because congruence in the K-geometry is different from congruence in the Euclidean plane where this interpretation takes place.

Our first notion of segment congruence is too meager. For one thing, we could never have a segment congruent to another segment on the same line. Thus we could not slide a K-length along a K-line, which makes it impossible to have rulers for measuring. So, we extend the definition to say that two K-segments are also congruent if they are congruent (in the sense of a perspectivity) to the same third segment.

We are now in the position of proving a part of the ruler axiom. We can show that K-lines are infinitely long by laying off infinitely many segments on a line in both directions so that all of them congruent to the same template segment.

Question 5 Let ℓ be a given K-line. It lies on an E-chord $P'Q'$. Chose a chord, $Q'R'$, and a segment QR on it. Pick the first viewpoint V_0 on the *base line*, $b = P'R'$, and transfer QR to it. But label the image Q_0Q_1, so that V_0, Q_0, Q are collinear. Now extend QQ_1 until it hits the base line at a new viewpoint, V_1. Repeat, i.e. transfer QR to ℓ using V_1. Note that $Q_0Q_1 \cong Q_1Q_2$ because both are congruent to QR. Be sure your draw sufficiently many pictures to be convincing.

Finish the proof by arguing that it is possible to continue this process forever in both directions. Can you discover an analogue in Euclidean geometry that proves the same theorem in the same way? Hint: Consider making the test segment QR parallel to ℓ and using parallel rays to measure out lengths equal to $|RQ|$ on ℓ.

Definition of K-congruence: K-segments $AB \cong CD$ if $A'ABB'$ is perspective to $C'CDD'$, where the primed letters refer to the endpoints of their chords, or, if there is a finite sequence of perspectivities beginning with AB and ending with CD.

The foregoing definition makes our congruence relation transitive as well as reflexive and symmetric[1].

Question 5.5 Don't just take that on faith. Prove that congruence, so defined, is an equivalence relation.

What we do *not* yet know is whether this definition is reasonable. For, if starting from AB we proceed by a sequence of perspectivities to a segment CD on K-line $C'D'$, and a different sequence of perspectivies brings us to CE also lying on the K-ray CD', then we fervently hope that $D = E$. But will this be true? We formulate exactly what we expect to be true.

Theorem. If one sequence of perspectivities transfers the K-segment AB to the K-ray CD' so that B goes to D, and another does so with B going to E then $D = E$.

This would seem to be very difficult to prove. And it is that unless we find a clever way around the brute force attempt of proving it. Imagine trying to say anything about two arbitrarily long sequences of constructions. One way to prove the theorem is to postulate a consistent way of measuring length. Then show that any perspectivity preserves length. After all, congruent segments should also be *isometric*, i.e. of the same length. Then $D = E$ because $\|CD\| = \|CE\|$ The segments have the same length.

[1]A relation $x \sim y$ on an abstract set $S = \{x, y, \ldots\}$ is said to be an *equivalence relation* if it is

Symmetric: $x \sim y \Rightarrow y \sim x$.

Reflexive: $x \sim x$.

Transitive: $x \sim y \,\&\, y \sim z \Rightarrow x \sim z$.

A congruence relation *partitions* its set into mutually exclusive, collectively exhaustive subsets, called the *equivalence classes*. For example, in Euclidean geometry, parallelism is an equivalence relation on the set of lines. An equivalence class here is called a 'pencil of parallels'. The Renaissance perspectivists, such as Albrecht Dürer, considered the collection of these so-called *ideal points* as an *ideal line* at infinity.

3.1 Perpendicular Bisectors

In Euclidean geometry, one of the first and most important constructions is to find the perpendicular bisector of a segment AB. Stick your compass into A and open it to B. Draw a large enough (¡180^o) arc. Perform the symmetric construction with A, B exchanged, and label the two points where the arcs cross, C, D. The line CD crosses AB at its midpoint, and it is perpendicular.

We could try and mimic this construction in a non-Euclidean context, but we would have to first have a non-Euclidean compass. Instead we invent a plausible construction, and then prove, either by example persuasively, or rigorously with analytic geometry, that it has the right properties.

3.2 Bisecting a K-segment.

Consider the segment AB on chord $A'B'$. To find the *K-midpoint* of AB proceed as follows:

(i) Locate where the tangents to the Kleindisk at A' and B' meet at P. This point is called the *pole of the chord* $A'B'$. An efficient way of constructing the tangent to a point R on a circle with center Q is to lay a right-angle so that its corner is on the circle, the center Q lies on one edge. Now draw the other edge. A transparent plastic triangle is good for this, but a filecard will do in a pinch.

(ii) Extend PA and PB to make two chords. Connect opposite endpoints of the these chords. Note that these "diagonals" cross at a point M that is also on AB.

In particular, henceforth you can just use one of the "diagonals" to find the midpoint.

Question 6 Verify this construction for several cases. Choose one that is particularly convincing. Start with a fairly long segment which is somewhat off-center. Copy the K-disk and AMB to a new drawing. Use a V-construction to copy AM to a chord that has one endpoint A' in common with the chord $A'B'$ and let D' be it's other endpoint. Find a convenient view point V on the baseline through $B'D'$ to look at AM through PQ on $A'D'$. Now, demonstrate that PQ occludes MB from a different viewpoint on the baseline.

Question 7 Estimate the locations of $\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}$ on the "unit" segment, AB.

Given a segment on a K-line to server as our "unit", it now is clear how, in principle, we can construct whole number multiples of it, both positive and negative. And, with more effort, we can find all binary fractions on all of these segments, in the K-geometry.

This isn't, yet, a complete ruler. For example, $\frac{2}{3}$ is not among the points we can construct. But we can approximate every real number by a sequence of binary fractions. So, "in the limit", we have placed all real numbers on the K-line.

The easiest way to see this is to identify the real numbers by their *binimal expansions*. Binimals work just like decimals, only easier. They extend the binary numeration to fractions. Recall that in binary numerals, we write 0=0, 1=1, but 2=10, and 3=11, etc. Past ten, we have notational problems and might write append the base to the numeral, for example $10_{10} = 1010_2$. (Do you believe that? On the left is says that you need one dime and no pennies to get ten cents. On the right, is says you need no penny, a twopenny, no fourpenny, and one eighpenny.)

Going the other way, we would write $\frac{1}{2}$ as the binimal 0.1, for exactly the same reason that in decimal we write it as 0.5. Convince yourself that $\frac{2}{3}$ is 0.101010... repeating.[2]

Thus, we can approximate every real number arbitarily closely by merely doubling and halving. This is called the *inchworm principle*. And, with it, we have verified one part of the *Ruler Axiom* in the Kleindisk, namely that every line supports (many) rulers. We don't know how to measure distance in an absolute sense, so we can't calibrate all these rulers to be consistent with a distance function. That will have to wait til later.

3.3 K-Perpendiculars

Euclid emphasized the importance of right angles and perpendicularity by postulating that "all right angles are equal." In the Beltrami-Klein model of the hyperbolic plane we define a line k to be perpendicular to a given line ℓ if and only if k passes throught he pole P of ℓ. From this construction it is not immediately obvious that this relation between two K-lines is *reflexive*.

Theorem If $k \perp_K \ell$ then $\ell \perp_K k$.

Proof. The simplest way to prove this would be to define perpendicularity in terms of a hyperbolic protractor to mean an angle of 90°. But the hyperbolic protractor is NOT easy to work with, while poles of lines ARE easy to construct with ruler and compass. However, a purely *synthetic* proof, i.e. one that uses only the classical argumentation familiar to all who have studied *Euclid's Elements* is also difficult. We would have to prove quite a few lemmas about poles and polars first. The *polar* of a point outside a circle is that chord of the circle whose pole is the given point. To construct the polar of P relative to K, draw the two tangents from P to K and the chord between the points of

[2]Need a hint? Recall how you showed in HS that the decimal X=0.333... is a third. Multiply it by 10 and subtract X to show that 9X=3. In binimals, multiply by four = 100 in the usual way and do the same algebra.

tangency. Instead, we shall use the Cartesian analytical geometry your learned in high-school.

Lemma 1. The pole (p, q) of the line $ax + by = c$ is given by the equations $p = a/c, q = b/c$. Note that lines through the center of K have their poles at infinity, and we need to treat such lines as special cases.

Question 8 Prove the Theorem synthetically for the case that ℓ is a diameter. Note that this case IS easy, because for such K-lines, K-perpendiculars are E-perpendiculars.

Proof of Lemma 1. The vector (a, b) is perpendicular to the line ℓ. Then for some value of the scalar $t, (ta, tb)$ lies on ℓ, and for some other value of t, you reach the pole. Substituting into the equation of the line, you will find that when $t = \frac{c}{a^2+b^2}$ you are at the *foot* [3] F of ℓ, Now the pole of the line is the inverse P of the foot F, which you can compute to be $(a/c, b/c)$.

Question 9 Given a circle K with center O, the *inverse* of a point P is the point Q on the ray OP whose distance OQ, measured in radial units of K, is the *reciprocal* of the distance OP. Compute that the inverse of (p, q) in the unit circle is $(\frac{p}{p^2+q^2}, \frac{q}{p^2+q^2})$.

Question 10 Now complete the computation that establishes the proof of Lemma 1. That is, write out the proof of this lemma completely, from start to finish.

Question 11 Note, in passing, that not just any equation $ax + by = c$ defines a chord of the unit circle K. Show that $a^2 + b^2 > c^2$ is necessary and sufficient for the line to pass through the circle. Hint, how far is the foot of the line from the origin?

Question 12 Explain why the equation of any K-perpendicular k to ℓ must have a formula that looks like this

$$\alpha(x - a/c) + \beta(y - b/c) = 0$$

and therefore its pole is $(\frac{\alpha c}{\alpha a+\beta b}, \frac{\beta c}{\alpha a+\beta b})$.

Question 13 What still has to be argued to finally prove the theorem? Here is how to answer such a question. Outline the proof of the theorem. Indicate chapter and verse (place in the notes and in previously answered questions) that supports each stepl. If any steps are incomplete, supply the argument in additional lemmas.

Comment. The foregoing is an elegant example of Cartesian geometry. Note how the step-by-step translation of geometric relationships into algebra not only simplifies the proofs, but even suggests how to proceed towards a proof. Verification of a geometric relationship is reduces to "solving" some equations.

Secondly, we have introduced the notion of *inversion. Inversive Geometry* is the systematic study of those geometrical properties that are invariant under this transformation in the plane. Euclidean geometry, then, might be defined as *the systematic study of those geometrical properties that are invariant under Euclidean transformations,* and so-on. Felix Klein's celebrated *Erlangen*

[3] The point on line which is nearest to a given point not on the line can be called its *foot* relative to that point. For chords of circles, the midpoint is their foot relative to the center.

Programm advocates defining a geometry in germs of the group of transformations, called its *isometries*, which are postulated to be its congruence criterion. We shall adopt the rudiments of inversive geometry as a convenient tool for understanding non-Euclidean geometry.