M13 – Symmetry

Note Title 4/19/2010

Motivation: The quiz on F13 covers weeks 12 and 11. So this week we'll discuss issues that reviews this work. Today we compare the geometric and algebraic methods for exploring the consequences of symmetry.

In this transcript of the lecture, green and red are commentaries. The geometrical approach, based on Thales' theory of similar right Δs, led to the formula $|z-p||z^*-p| = r^2$ for points symmetric relative to a circle centered at p of radius r.

From this we can "read" the figures for the first 4 properties but (5) is too difficult to prove geometrically.

The first colored figure suggests that $\overbrace{\lim_{z \to \text{mirror}} z^* = z}^{\text{green}}$ & $\overbrace{\lim_{z \to 0} z^* = \infty}^{\text{red}}$

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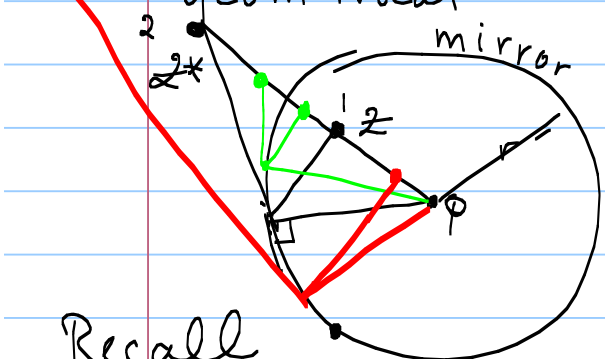
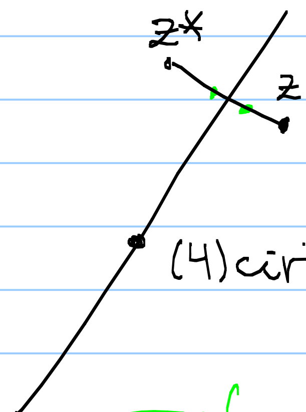
Inversions in Circles = Symmetries = Reflections in circlines

Geometrical





(1) $z^{**} = z$

(2) $z \in$ mirror $\Rightarrow z^* = z$

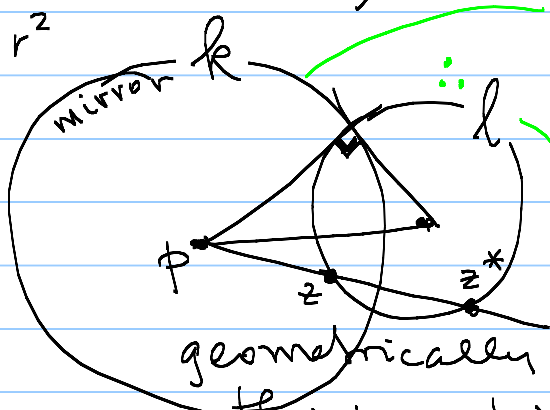
(3) $c =$ center $\Rightarrow c^* = \infty$

(4) circlines $\ell \perp k \Rightarrow k \perp \ell$

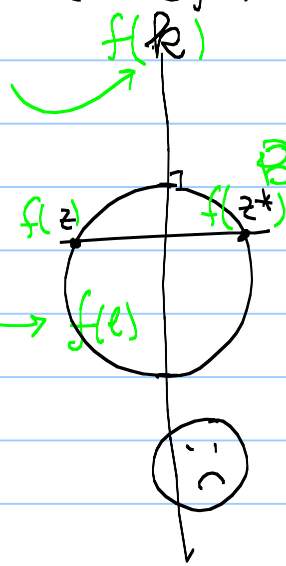
(5) If $\ell \perp k$ then $\ell^* = \ell$.

Recall

$|z-p||z^*-p| = r^2$



$f \in$ moeb



geometrically this is nontrivial

algebraically it is trivial

Möbius Transformations

Big Theorem

If $f(z) \in$ Möb then

(a) f takes cirlines to circlines

(b) symmetric points to symmetric points

Comment after class on the previous panel

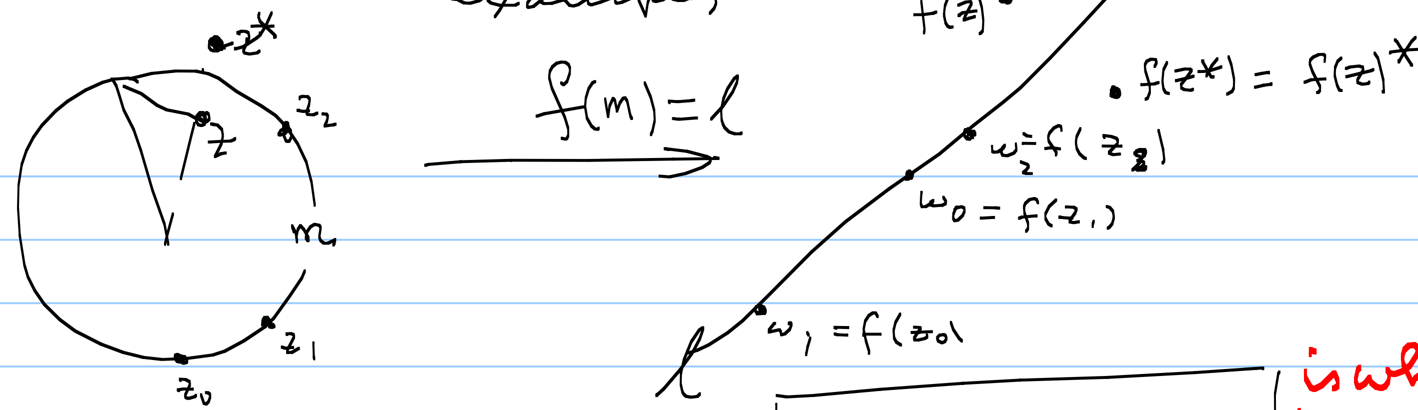
The Big Symmetry Theorem (BST) is useful to prove (S) as follows.

Given circle k and circle $l \perp k$ as on the left. Choose some $f \in$ möb with $f(p) = \infty$, then $f(k)$ is a straight line, and $f(l)$ is a 'circline' and $f(l)$ is still $\perp$ to $f(k)$.

It can't look like . Now it is clear that $f(l)$ is symmetric in the classical sense. The BST also says that if z, z^* are symmetric rel to k, $f(z)$ and $f(z^*)$ are symmetric rel to $f(k)$. Since $f(l)$ is symmetric across the line $f(k)$, its preimage l is symmetric across k.

The next panel just illustrates what I just said. Since the meaning of the $*$ notation depends on a context, let's replace it by placing the <u>name</u> of the mirror of symmetry.

examples



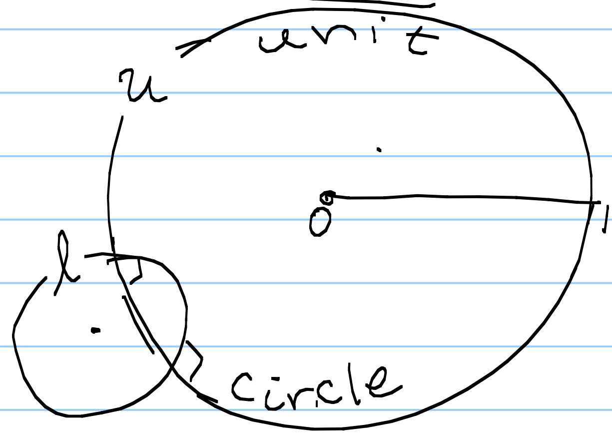
if we write $z^* = z^m = S_m(z)$ then $\boxed{f(z^m) = f(z)^{f(m)}}$ is what we have proved

i.e. for $w := f(z)$ and $f(k) := h$ then $f(z^m) = w^h$

Comment: Given circline k and circline $l \perp k$ then k is symmetric rel. to l (4 figures)

Pf: Send k to straight line $f(k)$ by some Moeb. Then $f(l)$ is a circline $\perp$ to line $f(k)$ where it is geometrically obvious.

Another example



What is the equation of a circle $\perp$ unit circle?

Recall

$|z+\mu|^2 = r^2$ is the equation of circ$(-\mu, r)$

i.e. centered at $-\mu$ and radius r.

Calculate

$$(z+\mu)(\bar{z}+\bar{\mu}) = r^2$$

$$z\bar{z} + \mu\bar{z} + \bar{\mu}z + |\mu|^2 = r^2$$

$$(*) \quad z\bar{z} + \mu\bar{z} + \bar{\mu}z + (|\mu|^2 - r^2) = 0$$

⚠ If the circle is symmetric w.r.t. the unit circle then its equation must be $\boxed{z\bar{z} + \mu\bar{z} + \bar{\mu}z + 1 = 0}$

pf: Subst $\frac{1}{\bar{z}}$ into $(*)$ $\quad \frac{1}{\bar{z}} \cdot \frac{1}{z} + \mu\frac{1}{z} + \bar{\mu}\frac{1}{\bar{z}} + (\delta) = 0$

multiply $z\bar{z}$ $\quad 1 + \mu\bar{z} + \bar{\mu}z + (\delta)z\bar{z} = 0$

which is the original equ^n iff $\delta = 1$

Elaboration:

The circle in question is $\{z \mid z\bar{z} + \mu\bar{z} + \bar{\mu}z + \delta = 0\} = \ell$ (where $\delta := |\mu|^2 - r^2$) and we say it is $\perp u$.

So, if $z \in \ell$ then $z^* \in \ell$ as well. So it must satisfy the equation. Since $z^* = z^u = \dfrac{1}{\bar{z}}$, substituting we find

$1 + \mu\bar{z} + \bar{\mu}z + \delta z\bar{z} = 0$ for every $z \in \ell$. The only way the two equations can be for the same circle is that $\delta = 1$.

Conversely, the circle $z\bar{z} + \mu\bar{z} + \bar{\mu}z + 1 = 0$ is $\perp u$ because the equation is invariant under the symmetry $z \to z^* = 1/\bar{z}$.