Lesson on Logic and Truth Tables

Revised for MA402 7jan15.

Content: In this lesson we review the elements of symbolic logic we need from the prerequisite course, MA347. In particular, recall the definitions, notation and concepts of **Propositional Calculus**. These can be checked by means of their truth tables.

Definition A *proposition* is a sentence which can be either true or false. If you cannot imagine discovering or assigning the truth value to the sentence, it is not a proposition.

Definition A *sentence* consists of a *subject* and a *predicate*. What a subject and predicate is we shall leave to your experience of 12 years of schooling.

Examples Here are three examples of sentences that are not proposition, but for different reasons.

- The king of France is bald.
- The study of square circles is difficult.
- Math class is tough.

Comment: The first non-proposition in the list is a classical example proposed by the most famous logician in recent times, Bernard Russell. Because France has not had a king for three centuries now, we cannot even theoretically check its truth.

The second example fails the criterion because it contains a impossible subject. There could be king of France, but there cannot be a square circle.

The third example is what one edition of Barbie Dolls[1] was programmed to say some years ago. Barbie's maker Mattel had to remove this sexist opinion by popular demand. Personal opinions are neither true nor false.

The logic of propositions:

We next develop the rudiments of *propositional logic*, which is a set of rules

[1] `http://www.youtube.com/watch?v=NOOcvqT1tAE`

concerning propositions. The rules of logic seem to be ingrained in our minds, humans think logically by nature. However, once a certain complexity sets in, for example in legal matters, or in mathematics, common sense logic is no longer reliable. We need rules. The establishment of rules in logic is an example of *formalization*. We formalize rules when we want an efficient and reliable way of teaching these rules to hitherto innocent students (or ignorant computers !).

Whenever we formalize a science, we need a language to do it with. The language we use to formalize something, is called a *meta-language*. Thus the rules of the propositional calculus themselves are the subject of *meta-logic*. But for present purposes the meta-logic can be *informal*, i.e. we will depend on common English and common sense to talk about the rules of logic.

Conjunction: The conjunction of two propositions A and B, is written $A \wedge B$ and is spoken “A and B”. To see that a conjunction is again a proposition, all we have to do is to decide the truth-values of the conjuction as a function of the truth-values of the the constituents.

We assign the symbol 0 for *false* and the symbol 1 for *true*. Then, a *truth table* for conjunctions looks like a multiplication table. [2]

$A \wedge B$	0	1
0	0	0
1	0	1

Another, more compact way of writing the same truth table could be

AB	00	01	10	11
$A \wedge B$	0	0	0	11

Logical equivalence: The symmetry of the and-table shows that the conjunction $B \wedge A$ has the same truth values as $A \wedge B$. Therefore, we consider these two propositions to be logically *equivalent* and write

$$A \wedge B \sim B \wedge A$$

[2] In fact, it is the multiplication table for the finite field of order 2 which you studied in MA347.

There are two further common notations used for logical equivalence.

$$A \wedge B \equiv B \wedge A \text{ and } A \wedge B \Leftrightarrow B \wedge A$$

.

Negations: The negation of a proposition A, written $\neg A$ (and sometimes as $\overline{A}$) , is also a proposition because it has a truth table which is

A	0	1
$\neg A$	1	0

Disjunction: The disjunction of two propositions A and B, is written $A \vee B$ and is spoken "A or B". To see that this is again a proposition, all we have to do is to decide the truth-values of the disjunction as a function of the truth-values of the the constituents.

$A \vee B$	0	1
0	0	1
1	1	1

Unlike with "and", we cannot depend on our common language here to distinguish "or" from "and/or" . Latin does have two different words for "or". The inclusive "and/or" is "vel" in Latin, and hence the symbol $\vee$. The exclusive or, also called "xor" and sometimes denoted by $\times$, has the Latin word "aut" for it.

Note the difference in their truth tables:

AB	00	01	10	11	
$A \vee B$	0	1	1	1	or
$A \times B$	0	1	1	0	xor
$A \equiv B$	1	0	0	1	equivalence

The last line illustrates an important observation that negating a proposition is the same as flipping its truth values. So the negation of "either A or B but not both are true" becomes "Both A and B are true or false together". Ask yourself, which disjunction applies to the word "or" in the previous quotation?

For just two constituent propositions there are 16 possible truth tables. As we just saw, they come in pairs, one being the negation of the other.

Implication: The most common, and most commonly misunderstood logical form in mathematics is *material implication*. You have met it in such statements as "If A then B", written $A \Rightarrow B$. Its truth table is not intuitively obvious, since it is *defined* to be

AB	00	01	10	11	
$A \Rightarrow B$	1	0	1	1	implies
$\neg A \vee B$	1	0	1	1	does not imply

Note that we have written down the same truth table twice, which proves that

$$A \Rightarrow B \Leftrightarrow \neg A \vee B$$

.

You must not forget this rule of propositional logic for the rest of this course. So, to verify a rule of propositional logic we may use previously proved rules. Or compute their truth tables and see how they match.

Exercises

1. Prove that $A \wedge (B \vee C)) \sim (A \wedge B) \vee (A \wedge C)$.
2. What is $A \vee (B \wedge C))$ equivalent to?
3. Simplify $\neg(A \Rightarrow B)$.
4. Are you surprised that the answer to 3. is not $B \Rightarrow A$?
5. Write " A if and only if B" in symbols and find its truth table.